The Hodge Decomposition

By Jordan Grant

December 7, 2020

1 Preliminaries

The purpose of this paper will be to give a proof of the Hodge Decomposition, along with a sneak peak into one of its applications, Poincare Duality. In order to do this, we begin with some preliminary definitions. For the sake of brevity we also assume that the reader has had some introduction to smooth manifold theory, and will introduce the bare minimum of components needed for our construction.

Definition: Have M be an oriented compact n-dimensional smooth Riemannian manifold with inner product $\langle\ ,\ \rangle$ given by the metric, and consider the tangent space T_pM for any $p \in M$. The Riemannian metric induces an inner product on the space of k-forms, $\Lambda^k T_p^* M$. For a k-form α, its associated **Hodge Dual** $\star\alpha$ is the unique $(n-k)$ form satisfying $\beta \wedge \star\alpha = \langle \beta, \alpha\rangle \omega$ where ω is the volume form and β is any given k-form. We then define the **Hodge Star Operator** as

$$\begin{gathered} \star : \Lambda^k T_p M \to \Lambda^{n-k} T_p M \\ \alpha \mapsto \star\alpha \end{gathered}$$

Integrating both sides of the equation $\beta \wedge \star\alpha = \langle \beta, \alpha\rangle\omega$ yields, $\int_M \beta \wedge \star\alpha = (\beta|\alpha)$, the inner product on k-forms.

Definition: Have (M, g) be an oriented n-dimensional smooth-Riemannian manifold with metric tensor g. For each k with $0 \leq k \leq n = \dim M$, the **codifferential** $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ is defined by

$$\delta := (-1)^{\text{ind}(g)}(-1)^{n(k+1)+1} \star d\star$$

where d is the exterior derivative. An important aspect of the codifferential is that it is the adjoint of the exterior derivative, which means that for α a k-form and β a $k+1$-form, $(\alpha|\delta\beta) = (d\alpha|\beta))$.

Remark: Note that our inner product $(\,|\,)$ is only guaranteed to be defined on our manifold due to our assumption of compactness. Due to this inner product being essential in the proof of the Hodge Decomposition, it follows that the Hodge Decomposition only holds true on compact manifolds.

Definition: We can now define the main player in the Hodge Decomposition, the **Hodge Laplacian**. The Hodge Laplacian is defined as

$$\Delta = d\delta + \delta d : \Omega^k(M) \to \Omega^k(M).$$

A differential k-form α is said to be ***harmonic*** if $\Delta\alpha = 0$. We denote the space of harmonic k-forms on our manifold as $\mathcal{H}^k(M)$.

Definition: For $\alpha \in \Omega^p(M)$ we define a **weak solution** of $\Delta\omega = \alpha$ as a bounded linear function $\ell : \Omega^k(M) \to \mathbb{R}$ such that

$$\ell(\Delta\beta) = \langle \alpha, \beta\rangle \text{ for } \forall\beta \in \Omega^p(M).$$

To prove the Hodge Decomposition we will need a couple of important facts about the Hodge Laplacian, as well as an important theorem. Their proofs are outside the scope of the prerequisites needed to understand the Hodge Decomposition, so we will proceed without proof. For those interested, their proofs can be found in **[1]**.

Regularity of the Hodge Laplacian:
Let $\alpha \in \Omega^p(M)$ and let ℓ be a weak solution of the equation $\Delta\omega = \alpha$. Then there exists $\omega \in \Omega^p(M)$ such that

$$\ell(\beta) = \langle \omega, \beta \rangle$$

for all $\beta \in \Omega^p(M)$. Therefore ℓ determines a solution of $\Delta\omega = \alpha$.

The Hodge Laplacian is a compact operator:
Let $\{\alpha_n\}$ be a sequence in $\Omega^k(M)$ such that $||\alpha_n|| \leq c$ and $||\Delta\alpha_n|| \leq c$ for all n and for some $c > 0$. Then there exists a subsequence $\{\alpha_{n_k}\}$ which is Cauchy in $\Omega^k(M)$.

Hahn-Banach Theorem: Let X be a real or complex normed linear space, let $M \subset X$ be a linear subspace, and let $\ell \in M^*$ be a bounded linear functional on M. Then there exists a linear function $\tilde{\ell} \in X^*$ that extends ℓ and satisfies $||\tilde{\ell}||_{X^*} = ||\ell||_{M^*}$.

2 The Hodge Decomposition

Let (M, g) be compact and oriented (and without boundary). For each k with $0 \leq k \leq n = \text{dim}M$, the space harmonic k-forms $\mathcal{H}^k$ is finite dimensional.

Proof:
If $\mathcal{H}^k$ were infinite-dimensional, then it would contain an infinite orthonormal sequence $\{\omega_i\}_1^\infty$, namely its basis. In this case we would have

$$||\omega_i - \omega_j||^2 = 2 \text{ for all } i, j \text{ with } i \neq j.$$

Seeing as $\{\omega_i\}_1^\infty$ is an orthonomoral basis it satisfies $||\omega_i|| \leq 1$ for all i, and seeing as $\omega_i \in \mathcal{H}^k$ for all i, it follows that $||\Delta\omega_i|| = ||0|| \leq 1$. Therefore by compactness of the Hodge Laplacian, there exists a Cauchy subsequence of our basis, implying that for some $n_1, n_2 \in \mathbb{N}$ $||\omega_{n_1} - \omega_{n_2}|| < 2$, a contradiction. $\square$

In order to prove the Hodge Decomposition we will be in need of a useful lemma. For the sake of brevity we will assume the lemma without proof, however the proof the lemma can be found in **[1]**, pg. 421.

Lemma: There exists a constant $C > 0$ such that $||\beta|| \leq C||\Delta\beta||$ for all $\beta \in (\mathcal{H}^k(M))^\perp$.

The Hodge Decomposition

Let (M, g) be compact and oriented (and without boundary). For each k with $0 \leq k \leq n = \text{dim}M$, we have an orthogonal decomposition of $\Omega^k(M)$,

$$\Omega^k(M) = \Delta(\Omega^k(M)) \oplus \mathcal{H}^k$$

$$= d\delta(\Omega^k(M)) \oplus \delta d(\Omega^k(M)) \oplus \mathcal{H}^k$$

$$= d(\Omega^{k-1}(M)) \oplus \delta(\Omega^{k+1}(M)) \oplus \mathcal{H}^k.$$

Proof:
Have $\omega_1, ..., \omega_d$ for $\mathcal{H}^k$ If $\alpha \in \Omega^k(M)$, then we can express α in the following way

$$\alpha = \beta + \sum_{i=1}^{d} (\alpha|\omega_i)\omega_i$$

where $\beta \in (\mathcal{H}^k)^\perp$. This gives us an orthogonal decomposition $\Omega^k(M) = (\mathcal{H}^k)^\perp \oplus \mathcal{H}^k$. If we can show that $(\mathcal{H}^k)^\perp = \Delta(\Omega^k(M))$ then we will have the first line of the Hodge Decomposition. If $\Delta\alpha \in \Omega^k(M)$, then by definition $(\Delta\alpha|\omega) = (\alpha|\Delta\omega) = 0$ whenever $\omega \in \mathcal{H}^k$. Therefore $\Delta(\Omega^k(M)) \subset (\mathcal{H}^k)^\perp$. Now let $\alpha \in (\mathcal{H}^k)^\perp$, and define a linear functional ℓ on $\Delta^k(M)$ by

$$\ell(\Delta\theta) := (\alpha|\theta)$$

To see that ℓ is well defined, note that if $\Delta\theta_1 = \Delta\theta_2$, then $\Delta(\theta_1 - \theta_2) = 0$ which implies that $\theta_1 - \theta_2 \in \mathcal{H}^k$ and so $(\alpha|\theta_1) - (\alpha|\theta_2) = (\alpha|\theta_1 - \theta_2) = 0$. Next we show that ℓ is bounded. Let $\phi := \theta - H(\theta)$ where $H_k : \Omega_k(M) \to \mathcal{H}^k$ is the orthogonal projection. Then by our lemma we have

$$|\ell(\Delta\theta)| = |\ell(\Delta\phi)| = |(\alpha|\phi)| \leq ||\alpha||\,||\phi|| \leq C||\alpha||\,||\alpha\Delta\phi|| = C||\alpha||\,||\Delta\phi|| = C||\alpha||\,||\Delta\theta||.$$

By the Hahn-Banach theorem, the functional ℓ extends to a bounded functional $\tilde{\ell}$ defined on all of $\Omega^k(M)$, which is then a weak solution of $\Delta\omega = \alpha$. By regularity of the Hodge Laplacian, there is an $\omega \in \Omega^k(M)$ with $\Delta\omega = \alpha$. Therefore $\alpha \in \Delta(\Omega^k(M))$, which implies that $(\mathcal{H}^k)^\perp \subset \Delta(\Omega^k(M))$, and $(\mathcal{H}^k)^\perp = \Delta(\Omega^k(M))$.

To get the second line of the decomposition simply expand the Hodge Laplacian

$$\Delta(\Omega^k(M)) \oplus \mathcal{H}^k = (d\delta + \delta d)(\Omega^k(M)) \oplus \mathcal{H}^k =$$

$$d\delta(\Omega^k(M)) \oplus \delta d(\Omega^k(M)) \oplus \mathcal{H}^k$$

To get the third line we simply need to prove equality between the first and second terms of our second and third lines respectively. We will only prove it for the first term seeing as the proof for the second is identical.

$$d\delta\Omega^k(M) = d(\Omega^{k-1}(M)):$$

Here we define our operators as $d : \Omega^{k-1}(M) \to \Omega^k(M)$ and $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$. Since δ is the natural adjoint of d with respect to $(\,|\,)$, it follows that $\text{Im}\delta = \delta(\Omega^{k-1}(M)) = (\text{Ker})^\perp$. For the first inclusion suppose that $\mathbf{0} \neq f \in d(\Omega^{k-1}(M))$. Then $f = d\omega$ for some $\omega \notin \text{Ker}d$, and therefore $\omega \in (\text{Ker})^\perp = \delta(\Omega^k(M))$. Therefore $f \in d\delta(\Omega^k(M))$. Seeing as $\delta(\Omega^k(M)) \subset \Omega^{k-1}(M)$ by definition, we get that $d\delta(\Omega^k(M)) \subset (\Omega^{k-1}(M))$. Therefore $d\delta(\Omega^k(M)) = d(\Omega^{k-1}(M))$ $\square$

Using this proof then gives us our decomposition

$$\Delta(\Omega^k(M)) \oplus \mathcal{H}^k = (d\delta + \delta d)(\Omega^k(M)) \oplus \mathcal{H}^k =$$

$$d\delta(\Omega^k(M)) \oplus \delta d(\Omega^k(M)) \oplus \mathcal{H}^k =$$

$$d(\Omega^{k-1}(M)) \oplus \delta(\Omega^{k+1}(M)) \oplus \mathcal{H}^k \ \square$$

3 Poincare Duality

An important consequence of the Hodge Decomposition is Poincare Duality. Poincare Duality is defined as the relationship between the homology and cohomology groups of a closed, oriented manifold given by the family of isomorphisms $H^p(M;\mathbb{Z}) \cong H_{n-p}(M;\mathbb{Z})$. In the case of a smooth manifold we also get an isomorphism of De Rham Cohomology groups, $H^k_{DR} \cong H^{n-k}_{DR}$ which is induced by the Hodge Star operator.

This ability to pass between higher cohomology to lower homology groups gives us a wealth of information when studying intersections of submanifolds which intersect transversely, and in some cases the existence of certain submanifolds can be verified solely using information provided by Poincare Duality. As a motivating example, consider a closed, oriented, odd dimensional smooth manifold M. Since M is closed, it is compact, therefore the Hodge Decomposition holds which gives us Poincare Duality. A quick application of Poincare Duality tells us that $\chi(M) = 0$, which implies, among other things, the existence of a non-zero vector field on M.

4 References

1. *Manifolds and Differential Geometry*, *Jeffery M. Lee*, 2009

2. *The Hodge Theorem and The Bochner Technique: A Vanishingly Short Proof*, *Hadrian Quan*,2015

3. Hand wiki: Hodge Star Operator https://handwiki.org/wiki/Hodge_star_operator